The Orbifolds of Permutation-Type as Physical String Systems at Multiples of c = 26 IV. Orientation Orbifolds Include Orientifolds

M. B. Halpern*
Department of Physics, University of California,
and Theoretical Physics Group,
Lawrence Berkeley National Laboratory
University of California, Berkeley, California 94720, USA

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Abstract

In this fourth paper of the series, I clarify the somewhat mysterious relation between the large class of *orientation orbifolds* (with twisted open-string CFT's at $\hat{c} = 52$) and *orientifolds* (with untwisted open strings at $c = 26$), both of which have been associated to division by world-sheet orientation-reversing automorphisms. In particular – following a spectral clue in the previous paper – I show that, even as an *interacting string system*, a certain half-integer-moded orientation orbifold-string system is in fact equivalent to the archetypal orientifold. The subtitle of this paper, that orientation orbifolds include and generalize standard orientifolds, then follows because there are many other orientation orbifold-string systems – with higher fractional modeing – which are not equivalent to untwisted string systems.

*halpern@physics.berkeley.edu

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1. Introduction

Non-abelian and free-bosonic *orientation orbifolds* [12,13,15] were first studied as a large class of examples in the orbifold program [1-11,12-15], which attempts to construct all orbifold CFT's. From the beginning there has been some mystery about the relation of the orientation orbifolds to ordinary *orientifolds* [16], since both classes ostensibly arise in the division $A(H_-)/H_-$ of a CFT $A(H_-)$ by a symmetry group H_- which contains world-sheet orientation-reversing automorphisms. However, following standard methods in orbifold theory, the orientation-orbifold CFT's contain twisted open-string sectors (corresponding to the orientation-reversing automorphisms) at *twice* the central charge c of $A(H_-)$ – as well as an equal number of twisted closed-string sectors (which form an ordinary space-time orbifold) at c. By twisted I mean fractionally-moded, as expected in the twisted sectors of any orbifold CFT. A simple understanding of the doubled central charge of the twisted open-string sectors is found in Ref. [12].

In the present series of papers [17-19], we have been considering the critical orbifolds of permutation-type as candidates for new *physical string systems* at higher central charge, including the space-time permutation orbifolds at any multiple of $c = 26$, as well as the general free-bosonic orientation orbifold:

$$\frac{U(1)^{26}}{H_-}, \quad H_- = \mathbb{Z}_2(\text{world sheet}) \times H \tag{1.1}$$

In these cases, the non-trivial element of $\mathbb{Z}_2(\text{w.s.})$ permutes the left- and right-movers of the critical closed string $U(1)^{26}$ while the extra automorphisms H act uniformly on both chiralities. The twisted open-string sectors of these orbifold CFT's show an essentially-unbounded variety of fractional modeing at central charge $\hat{c} = 52$.

As *string theories* however, the orbifolds of permutation-type must also satisfy certain *extended physical-state conditions* [16-18], which restrict their physical spectra relative to the underlying orbifold CFT's, and this has led to some surprises. In particular, although the $\hat{c} = 52$ orbifold-string spectra [18] are generically unfamiliar, we have learned that the spectra of some of the simplest (half-integer-moded) closed or open $\hat{c} = 52$ strings are equivalent to those of ordinary untwisted closed or open strings at $c = 26$.

In the present paper, I will follow one of these spectral clues to clarify the relation between orientation orbifolds and orientifolds.

1) In the first place, I will study here only the simple two-sector, half-integer-moded orientation orbifold

$$\frac{U(1)^{26}}{H_-}, \quad H_- = (1; \tau_- \times (-1)) \tag{1.2}$$

in further detail, showing that – even as an *interacting string theory* – this orientation orbifold is equivalent to the archetypal orientifold

$$\sigma = 0: \quad \text{unoriented closed string at } c = 26 \tag{1.3a}$$

$$\sigma = 1: \quad \text{twisted open string at } \hat{c} = 52 \tag{1.3b}$$

$$= \text{ordinary } NN \text{ string at } c = 26$$

that is, the conventional open-closed bosonic string system [20]. In the course of this discussion, we will see the extended physical-state conditions of Refs. [17-19] realized via *extended Ward identities* in the twisted-tree diagrams of the orbifold-string system – and our unconventional, orientation-orbifold description of the conventional system also gives a new and very simple derivation of the ratio of open- to closed-string Regge slopes.

2) A second conclusion borders on the philosophical: In contrast with the language of orientifolds , our construction clearly shows the relation of the conventional open-closed string system to standard orbifold conformal field theory [21-25,1-15]. In particular, the twisted sectors of any orbifold CFT must show fractional modeing, and indeed – although it is not visible in mass-shell emission from the boundaries – half-integer modeing is still present in the bulk of our twisted open string (1.3b).

3) The final conclusion is that, as string theories, orientation orbifolds include orientifolds

$$\{\text{orientation orbifolds}\} \supset \{\text{orientifolds}\}. \tag{1.4}$$

This embedding follows from the equivalence discussed here for the half-integer-moded case (1.2), and the fact that there exist many other orientation orbifolds (shown in Eq. (1.1)) with higher fractional modeing whose critical orbifold-string systems are *not equivalent* [19] to untwisted strings.

It is hoped that the computations given here will serve as a prototype for studying these more general orbifold-string systems at the interacting level. I remind that all the $\hat{c} = 52$ orbifold-string spectra have an equivalent, unconventionally-twisted $c = 26$ description [19] of the twisted $\hat{c} = 52$ matter. Indeed, our result here extends an example of this spectral equivalence to

the interacting level, and the general $c = 26$ spectral equivalence strongly suggests that an equivalent but generically-unconventional $c = 26$ description may also exist for all these interacting theories.

2. A Simple Orientation Orbifold

To establish notation, I begin with a few well-known facts about the ordinary (decompactified) critical closed bosonic string $U(1)^{26}$:

$$L(0) = -\frac{J^2(0)}{2} + R, \qquad \bar{L}(0) = -\frac{\bar{J}^2(0)}{2} + \bar{R} \tag{2.1a}$$

$$\bar{R} = R, \qquad J_\mu(0) = \bar{J}_\mu(0) \simeq T_\mu, \qquad \mu = 0, 1, \ldots, 25 \tag{2.1b}$$

$$A \cdot B = A_\mu \eta^{\mu\nu} B_\nu, \qquad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{2.1c}$$

Here $\eta^{\cdot}$ is the inverse target-space metric, the zero-mode eigenvalues $\{T\}$ are the dimensionless momenta and the relations in Eq. (2.1b) are consequences of level-matching $L(0) = \bar{L}(0)$. The relation to the dimensionful momenta $\{k\}$ is

$$T_\mu = \sqrt{\alpha'_c}\, k_\mu \tag{2.2}$$

where α'_c is the ordinary closed-string Regge slope. To eliminate negative-norm states, one must fix $L(0) = \bar{L}(0) = 1$ and hence $T^2 = -2$ for the closed-string ground state, but I will relax this mass-shell condition during our discussion of the orientation orbifold as a conformal-field-theoretic system.

The particular orientation orbifold we will study in detail is

$$\frac{U(1)^{26}}{H_-}, \quad H_- = (1;\, \tau_- \times (-1)) \tag{2.3}$$

where τ_- is world-sheet orientation reversal. The two sectors σ of this orbifold correspond to the following automorphic action on the untwisted current modes of $U(1)^{26}$

$$\sigma = 0: \quad J_\mu(m)' = J_\mu(m), \quad \bar{J}_\mu(m)' = \bar{J}_\mu(m) \tag{2.4a}$$

$$\sigma = 1: \quad J_\mu(m)' = -\bar{J}_\mu(m), \quad \bar{J}_\mu(m)' = -J_\mu(m) \tag{2.4b}$$

and the corresponding orientation-orbifold sectors, constructed by standard orbifold methods from $U(1)^{26}$, are similarly labeled:

$$\begin{aligned} &\sigma = 0: \quad \text{untwisted (symmetric) sector at } c = 26 \\ &\sigma = 1: \quad \text{twisted sector at } \hat{c} = 52. \end{aligned} \tag{2.5}$$

In particular, the untwisted sector is an unoriented closed string, while the twisted sector is the half-integer-moded open-string conformal field theory discussed in the following section.

3. The Twisted Sector as an Open-String CFT

The twisted open-string CFT of the orientation orbifold (2.3) was constructed from the closed-string sector {untwisted closed $\to$ twisted open} among the large class of free-bosonic examples in Refs. [12,13,15], and further discussed in Refs. [17,19]. More specifically, we may use the metric relation $G_{\mu\nu} = -\eta_{\mu\nu}$ to read off many results from Ref. [15], beginning with the action, the mode expansions and the orbifold Virasoro generators of this particular twisted sector:

$$\hat{S} = -\frac{1}{4\pi}\eta^{\mu\nu}\int dt \int_0^\pi d\xi \sum_{u=0}^{1} \partial_+ \hat{x}^{1\mu u}\, \partial_- \hat{x}^{1\nu,-u} \tag{3.1a}$$

$$\partial_\pm = \partial_t \pm \partial_\xi \tag{3.1b}$$

$$\mathrm{DN}: \quad \hat{x}_{1\mu 0}(\xi,t) = 2\sum_{m\in\mathbb{Z}} \frac{\hat{J}_{1\mu 0}(m+\frac{1}{2})}{m+\frac{1}{2}} e^{-i(m+1/2)t} \sin\left((m+\tfrac{1}{2})\xi\right) \tag{3.1c}$$

$$\mathrm{NN}: \quad \hat{x}_{1\mu 1}(\xi,t) = \hat{q}_{1\mu 1} + 2\hat{J}_{1\mu 1}(0)t + 2i\sum_{m\neq 0} \frac{\hat{J}_{1\mu 1}(m)}{m} e^{-imt}\cos(m\xi) \tag{3.1d}$$

$$\begin{aligned}\hat{L}_u\left(m+\tfrac{u}{2}\right) &= \tfrac{13}{8}\delta_{m+\frac{u}{2},0} \\ &\quad - \tfrac{1}{4}\eta^{\mu\nu}\sum_{v=0}^{1}\sum_{p\in\mathbb{Z}} :\hat{J}_{1\mu v}\left(p+\tfrac{v+1}{2}\right)\hat{J}_{1\nu,u-v}\left(m-p+\tfrac{u-v-1}{2}\right):_M \,.\end{aligned} \tag{3.1e}$$

In these results, $:\cdot:_M$ is standard mode normal-ordering, all quantities are periodic $u \to u \pm 2$ and $\bar{u} = 0,1$ is the pullback to the fundamental region. Up and down indices are simply related

$$\hat{A}^{\mu u} = \tfrac{1}{2}\eta^{\mu\nu}\hat{A}_{\nu,-u} = \tfrac{1}{2}\eta^{\mu\nu}\hat{A}_{\nu u} \tag{3.2}$$

where the last form holds by periodicity in u. Note that this open-string sector contains 26 integer-moded NN degrees of freedom and 26 half-integer-

moded [1] DN degrees of freedom, giving a total central charge $\hat{c} = 52$. Half-integer moding is of course the standard twisting of an order-two automorphism such as that in Eq. (2.4b), and a simple understanding of the doubled central charge of all open-string orientation-orbifold sectors is given in Ref. [12].

The twisted algebras [2] of this sector are

$$\left[\hat{J}_{1\mu u}\left(m+\tfrac{u+1}{2}\right), \hat{J}_{1\nu v}\left(n+\tfrac{v+1}{2}\right)\right] = -2\left(m+\tfrac{u+1}{2}\right)\eta_{\mu\nu}\delta_{m+n+\frac{u+v}{2}+1,0} \tag{3.3a}$$

$$\left[\hat{L}_u(m+\tfrac{u}{2}), \hat{J}_{1\mu v}(n+\tfrac{v+1}{2})\right] = -\left(n+\tfrac{v+1}{2}\right)\hat{J}_{1\mu,u+v}\left(m+n+\tfrac{u+v+1}{2}\right) \tag{3.3b}$$

$$\begin{aligned}\left[\hat{L}_u\left(m+\tfrac{u}{2}\right), \hat{L}_v\left(n+\tfrac{v}{2}\right)\right] &= (m-n+\tfrac{u-v}{2})\hat{L}_{u+v}\left(m+n+\tfrac{u+v}{2}\right) \\ &\quad + \tfrac{52}{12}\left(m+\tfrac{u}{2}\right)\left((m+\tfrac{u}{2})^2-1\right)\delta_{m+n+\frac{u+v}{2},0}\end{aligned} \tag{3.3c}$$

where Eq. (3.3c), which appears universally in all $\hat{c} = 52$ orbifold strings, is called an order-two orbifold Virasoro algebra [1,26,9]. The generators $\{\hat{L}_0(m)\}$ of the integral Virasoro subalgebra, with central charge $\hat{c} = 52$, are the physical Virasoro generators of the sector. Note also the presence of two sets of time-like modes $\{\hat{J}_{10u}, \bar{u} = 0, 1\}$ and hence twice the conventional number of negative-norm states in the CFT. As string theories however, we know that all twisted open and closed $\hat{c} = 52$ strings must also satisfy the so-called *extended physical-state conditions* [18,19]

$$\left(\hat{L}_u\left((m+\tfrac{u}{2}) \geq 0\right) - \tfrac{17}{8}\delta_{m+\frac{u}{2},0}\right)|\chi\rangle = 0, \quad \bar{u} = 0, 1 \tag{3.4}$$

leaving a physical spectrum which in this particular case [19] is identical to that of an ordinary untwisted open NN string at $c = 26$. The extended physical-state conditions (3.4) are the operator-analogues of the extended Polyakov constraints of $\mathbb{Z}_2$-twisted permutation gravity [17] and, more precisely, these conditions are a consequence of the twisted BRST systems [18] of $\hat{c} = 52$ matter. We will see another derivation of these conditions for $(m + \frac{u}{2}) > 0$ at the interacting level below.

[1]Half-integer-moded scalar fields [21] (and the corresponding twisted open strings with ND or DN boundary conditions [22,23]) provided the first examples of twisted sectors of orbifolds.

[2]The full quasi-canonical algebra, branes and twisted non-commutative geometry of the general free-bosonic orientation orbifold is given in Refs. [12,15].

To study the interacting theory we will also need the twisted Minkowski-space vertex operator

$$\hat{\hat{g}}_+(T,\xi,t) \propto \; :e^{-\frac{i}{2}(\hat{x}_{1\mu 0}(\xi,t)+\hat{x}_{1\mu 1}(\xi,t))\eta^{\mu\nu}T_\nu}: \tag{3.5}$$

which is an example of the free-bosonic or abelian "limit" of the twisted affine primary fields [6-15] of the non-abelian orientation orbifolds. According to the orbifold construction {closed string → twisted open string} in Refs. [12,13,15], the quantities $\{T\}$ in Eq. (3.5) are the *same* untwisted closed-string momenta $\{T\}$ defined for the ordinary closed string $U(1)^{26}$ in Eqs. (2.1) and (2.2). The vertex operator given here is in fact only one of two irreducible components of the vertex operator derived in Refs. [12,15], the latter including an extra 2×2 reducible matrix structure which is further discussed in Appendix A.

I limit the discussion here to the twisted vertex operator for emission at $\xi=\pi$ (returning later to emission at $\xi=0$). Defining the quantities

$$z \equiv e^{it}, \quad A\cdot B = A_\mu \eta^{\mu\nu} B_\nu \tag{3.6}$$

we find that the precise form [3] of this emission vertex is

$$\hat{g}_+(T,\pi,z) \equiv \; z^{T^2}\hat{\hat{g}}_+(T,\pi,z) \tag{3.7a}$$

$$\begin{aligned} &= z^{T^2} e^{-i\sqrt{2}T\cdot q} e^{-\sqrt{2}\ln z\, T\cdot J(0)} \times \\ &\times \exp\left[-\sqrt{2}T\cdot\sum_{m\geq 1}\frac{J(-m)}{m}(-1)^m z^m\right] \exp\left[\sqrt{2}T\cdot\sum_{m\geq 1}\frac{J(m)}{m}(-1)^m z^{-m}\right] \\ &\qquad \times \exp\left[-iT\cdot\sum_{m\leq -1}\frac{\hat{J}_{10}(m+\frac{1}{2})}{m+\frac{1}{2}}(-1)^m z^{-(m+\frac{1}{2})}\right] \\ &\qquad\qquad \times \exp\left[-iT\cdot\sum_{m>0}\frac{\hat{J}_{10}(m+\frac{1}{2})}{m+\frac{1}{2}}(-1)^m z^{-(m+\frac{1}{2})}\right]. \end{aligned} \tag{3.7b}$$

For computational convenience, I have here reexpressed the integer-moded NN subsystem in terms of conventional Fubini-Veneziano operators [27]

$$J_\mu(m) \equiv \frac{1}{\sqrt{2}}\hat{J}_{1\mu 1}(m), \quad q_\mu \equiv \frac{1}{2\sqrt{2}}\hat{q}_{1\mu 1} \tag{3.8a}$$

$$[q_\mu, J_\nu(0)] = -i\eta_{\mu\nu}, \quad [J_\mu(m), J_\nu(n)] = -m\eta_{\mu\nu}\delta_{m+n,0} \tag{3.8b}$$

[3] See in particular Subsec. 5.4 of Ref. [15].

but I keep the original notation for the commutators with the vertex operator:

$$\left[\hat{J}_{1\mu u}\left(m+\tfrac{u+1}{2}\right),\hat{g}_+(T,\pi,z)\right]=e^{i\pi(m+\frac{u+1}{2})}2T_\mu z^{m+\frac{u+1}{2}}\hat{g}_+(T,\pi,z) \tag{3.9a}$$

$$\left[\hat{L}_u\left(m+\tfrac{u}{2}\right),\hat{g}_+(T,\pi,z)\right]=e^{i\pi(m+\frac{u}{2})}\left(z\partial_z+(m+1+\tfrac{u}{2})4\Delta(T)\right)\hat{g}_+(T,\pi,z) \tag{3.9b}$$

$$\Delta(T)\equiv-\frac{T^2}{2}. \tag{3.9c}$$

The phases in Eqs. (3.7) and (3.9) are a consequence of the choice $\xi=\pi$ in the twisted string cooordinates (3.1c,d).

Finally, we will need the following properties of the momentum-boosted twist-field states

$$|T\rangle\equiv\lim_{z\to 0}z^{-T^2}\hat{g}_+(T,\pi,z)|0\rangle \tag{3.10a}$$

$$\left(J_\mu(m\geq 0)-\sqrt{2}T_\mu\delta_{m,0}\right)|T\rangle=J_{1\mu 0}\left((m+\tfrac{1}{2})>0\right)|T\rangle=0 \tag{3.10b}$$

$$\left(\hat{L}_u\left((m+\tfrac{u}{2})\geq 0\right)-\left(\tfrac{13}{8}+2\Delta(T)\right)\delta_{m+\frac{u}{2},0}\right)|T\rangle=0 \tag{3.10c}$$

one of which will be selected below as the ground state of the twisted open string.

4. Twisted Tree Graphs

I turn now to discuss the twisted $\hat{c}=52$ open-string CFT of the previous section as a sector of the full *interacting* orientation-orbifold string theory.

Using the quantities introduced in the previous section, I define the n-point *twisted tree graphs* of the interacting string theory by the following "sidewise construction":

$$\begin{aligned}\hat{A}_n(\{T\})\equiv\langle -T^{(n)}|\hat{g}_+(T^{(n-1)},\pi,1)\hat{D}[\hat{L}_0(0)]\hat{g}_+(T^{(n-2)},\pi,1)\cdots\\ \cdots\hat{D}[\hat{L}_0(0)]\hat{g}_+(T^{(2)},\pi,1)|T^{(1)}\rangle\end{aligned} \tag{4.1a}$$

$$\hat{D}[\hat{L}_0(0)]=\frac{1}{2(\hat{L}_0(0)-\hat{a}_2)}=\frac{1}{2}\int_0^1 dx\,x^{\hat{L}_0(0)-\hat{a}_2-1} \tag{4.1b}$$

$$\hat{a}_2=\frac{17}{8}. \tag{4.1c}$$

Here $\hat{g}_+$ is the twisted vertex operator (3.7) for emission at $\xi = \pi$, now additionally evaluated at $z = 1$. In the twisted propagator $\hat{D}$, the operator $\hat{L}_0(0)$ is the zero mode of the integral Virasoro subalgebra, and the constant $\hat{a}_2$ was determined from the $m = u = 0$ component of the extended physical-state condition in Eq. (3.4). In the sidewise construction, one sees the $\hat{c} = 52$ twisted open-string sector running horizontally (sidewise) in the direct channel, whereas the behavior of the twisted trees in the cross channels is not yet visible [4].

Comparing the propagator $\hat{D}$ with the properties of the twist-field states in Eq. (3.10), one then finds that the ground state and hence the twisted vertex operators must satisfy the mass-shell conditions

$$\Delta(T^{(i)}) = \frac{1}{4}, \quad (T^{(i)})^2 = -\frac{1}{2}, \quad i = 1, \ldots, n \tag{4.2}$$

so that the ground state is a pole of the propagator. Moreover, these conditions and the commutator (3.9b) with the extended Virasoro generators imply the following further properties of the twisted vertex operators

$$\hat{g}_+(T,\pi,z) = z^{\hat{L}_0(0)}\hat{g}_+(T,\pi,1)z^{-\hat{L}_0(0)-1} \tag{4.3a}$$

$$\left[\left(e^{-i\pi(m+\frac{u}{2})}\hat{L}_u(m+\tfrac{u}{2}) - \hat{L}_0(0)\right), \hat{g}_+(T,\pi,1)\right] = (m+\tfrac{u}{2})\hat{g}_+(T,\pi,1) \tag{4.3b}$$

where Eq. (4.3b) is a generalization of the so-called stability condition [29] in untwisted string theory.

5. Extended Ward Identities at $\hat{\mathbf{c}} = 52$

It was conjectured in Ref. [18] that the extended physical-state conditions in Eq. (3.4) would also follow from extended Ward identities in the interacting theory.

To see this explicitly in the present example, I begin by defining the extended (twisted) gauge operators:

$$\hat{W}_u(m+\tfrac{u}{2}) \equiv e^{-i\pi(m+\frac{u}{2})}\hat{L}_u(m+\tfrac{u}{2}) - \left(\hat{L}_0(0) + m + \tfrac{u}{2} - \hat{a}_2\right) \tag{5.1a}$$

$$\bar{u} = 0, 1. \tag{5.1b}$$

[4]The sidewise construction may be unfamiliar today, though it was well-known in the first string era. Indeed it was this construction which was used to obtain the (sidewise) $R \to NS$ amplitudes [28] and, with the addition of twisted scalar fields, the (sidewise) twisted sector $\to$ untwisted sector amplitudes [21] in early orbifold theory.

In order to see that these gauges are active in the twisted trees (4.1), the following vertex and propagator identities are helpful

$$\hat{W}_u(m+\tfrac{u}{2})\hat{g}_+(T^{(i)},\pi,1) = \hat{g}_+(T^{(i)},\pi,1)\left(e^{-i\pi(m+\frac{u}{2})}\hat{L}_u(m+\tfrac{u}{2}) - \hat{L}_0(0) + \hat{a}_2\right) \tag{5.2a}$$

$$\left(e^{-i\pi(m+\frac{u}{2})}\hat{L}_u(m+\tfrac{u}{2}) - \hat{L}_0(0) + \hat{a}_2\right)\hat{D}[\hat{L}_0(0)] = \tag{5.2b}$$

$$= \hat{D}\left[\hat{L}_0(0) + m + \tfrac{u}{2}\right]\hat{W}_u(m+\tfrac{u}{2})$$

where Eqs. (5.2a) and (5.2b) follow respectively from the extended stability condition (4.3b) and the orbifold Virasoro algebra (3.3c).

Then we find after some algebra the *extended Ward identities* at $\hat{c}=52$:

$$\hat{W}_u\left((m+\tfrac{u}{2})>0\right)\hat{g}_+(T^{(m)},\pi,1)\hat{D}[\hat{L}_0(0)]\cdots\hat{D}[\hat{L}_0(0)]\hat{g}_+(T^{(1)},\pi,1)|\chi\rangle = 0 \tag{5.3a}$$

$$\forall\,|\chi\rangle \text{ s.t. } \left(\hat{L}_u\left((m+\tfrac{u}{2})\geq 0\right) - \hat{a}_2\delta_{m+\frac{u}{2},0}\right)|\chi\rangle = 0, \quad \bar{u}=0,1. \tag{5.3b}$$

It should be emphasized that Eq. (5.3b) is the *same* extended (twisted) physical-state condition (3.4) obtained from the general twisted $\hat{c}=52$ BRST system in Ref. [18]. In particular, Eqs. (3.10c) and (4.2) tell us that the twisted open-string ground state at $T^2=-1/2$ is a physical state:

$$|T\rangle = \lim_{z\to 0} z^{\frac{1}{2}}\hat{g}_+(T,\pi,z)|0\rangle, \quad \langle -T| = \lim_{z\to\infty}\langle 0|\,z^{\frac{3}{2}}\hat{g}_+(T,\pi,z) \tag{5.4a}$$

$$\left(\hat{L}_u\left((m+\tfrac{u}{2})\geq 0\right) - \hat{a}_2\delta_{m+\frac{u}{2},0}\right)|T\rangle = \langle -T|\left(\hat{L}_u\left((m+\tfrac{u}{2})\leq 0\right) - \hat{a}_2\delta_{m+\frac{u}{2},0}\right)$$

$$= 0. \tag{5.4b}$$

Unconventional prefactors in asymptotic conditions, such as those in Eqs. (3.10) and (5.4a), are well-known in the orbifold program (see e.g. Ref. [9]).

I also remind that the solution of the extended physical-state condition (5.3b) with the extended Virasoro generators (3.1e) is known [19], showing that the physical spectrum of this *particular* twisted open $\hat{c}=52$ string is the *same* as that of an ordinary untwisted open NN string at $c=26$. This leads us to suspect with Ref. [19] that the twisted trees (4.1) of the orientation orbifold may be an unconventional realization of the tree graphs of ordinary NN strings.

6. Evaluation of the Twisted Trees

To evaluate the twisted trees, I begin with the n-point correlators of the open-string orientation-orbifold CFT

$$\begin{aligned}&\langle 0|\hat{g}_+(T^{(n)},\pi,z_n)\cdot\cdots\hat{g}_+(T^{(1)},\pi,z_1)|0\rangle\\&\quad=\delta^{26}\left(\sum_{i=1}^{n}T^{(i)}\right)\prod_{j=1}^{n}z_i^{-\frac{1}{2}}\sum_{i<j}\left\{z_i(1-\frac{z_j}{z_i})\left(\frac{1-(z_j/z_i)^{1/2}}{1+(z_j/z_i)^{1/2}}\right)\right\}^{-2T^{(i)}\cdot T^{(j)}}\end{aligned}\tag{6.1a}$$

$$=\delta^{26}\left(\sum_{i=1}^{n}T^{(i)}\right)\prod_{j=1}^{n}z_i^{-\frac{1}{2}}\sum_{i<j}\left(\sqrt{z_i}-\sqrt{z_j}\right)^{-4T^{(i)}\cdot T^{(j)}}\tag{6.1b}$$

which follow from the twisted vertex operators (3.7) when $\forall\,\Delta(T^{(i)})=1/4$ (see Eq. (4.2). Then we may use the integral representation (4.1b) of the twisted propagator to evaluate the sidewise construction in Eq. (4.1) as follows:

$$\begin{aligned}\hat{A}_n(\{T\})&=2^{-(n-3)}\int_0^1\prod_{i=2}^{n-2}(dx_i\,x_i^{-\hat{a}_2-1})\times\\&\quad\times\langle -T^{(n)}|\hat{g}_+(T^{(n-1)},\pi,1)x_{n-2}^{\hat{L}_0(0)}\cdots x_1^{\hat{L}_0(0)}\hat{g}_+(T^{(2)},\pi,1)|T^{(1)}\rangle\end{aligned}\tag{6.2a}$$

$$\begin{aligned}&=2^{-(n-3)}\int_0^1 dz_{n-2}\int_0^{z_{n-2}}dz_{n-3}\cdots\int_0^{z_3}dz_2\times\\&\quad\times\lim_{\substack{z_n\to\infty\\z_{n-1}\to1\\z_1\to0}}z_1^{1/2}z_n^{3/2}\langle 0|\hat{g}_+(T^{(n)},\pi,z_n)\cdots\hat{g}_+(T^{(1)},\pi,z_1)|0\rangle\end{aligned}\tag{6.2b}$$

$$\begin{aligned}&=2^{-(n-3)}\delta^{26}\left(\sum_{i=1}^{n}T^{(i)}\right)\int_0^1 dz_{n-2}\int_0^{z_{n-2}}dz_{n-3}\cdots\int_0^{z_3}dz_2\times\\&\times\prod_{i=2}^{n-2}z_i^{-2T^{(i)}\cdot T^{(1)}-\frac{1}{2}}(1-\sqrt{z_i})^{-4T^{(n-1)}\cdot T^{(i)}}\prod_{i<j}\left(\sqrt{z_i}-\sqrt{z_j}\right)^{-4T^{(i)}\cdot T^{(j)}}\ .\end{aligned}\tag{6.2c}$$

Here I have used the boost (4.3a) and the asymptotic relations (5.4a), as well as the standard change of variables $\{z_i\equiv\prod_{j=i}^{n-2}x_j\}$ to obtain Eq. (6.2b)

– and the correlators (6.1) to obtain the last form. The roots in all these expressions reflect the half-integer modeing of the open-string orientation-orbifold CFT.

But now consider the non-linear change of variables

$$z_i \equiv u_i^2, \quad i = 2, \ldots, n-2 \tag{6.3}$$

which puts our result in the final form:

$$\hat{A}_n(\{T\}) = \delta^{26}\left(\sum_{i=1}^{n} T^{(i)}\right) \int_0^1 du_{n-2} \int_0^{u_{n-2}} du_{n-3} \cdots \int_0^{u_3} du_2 \times$$
$$\times \prod_{i=2}^{n-2} \left(u^{-4T^{(i)}\cdot T^{(1)}} (1-u_i)^{-4T^{(n-1)}\cdot T^{(i)}}\right) \prod_{i<j} (u_i - u_j)^{-4T^{(i)}\cdot T^{(j)}}. \tag{6.4}$$

The non-linear transformation has removed all the roots, and indeed, under the following identification of the ordinary, untwisted open-string Regge slope α'_0 and the dimensionful momenta $\{k\}$

$$\sqrt{2}T^{(i)} = \sqrt{\alpha'_0} k_i, \quad \alpha'_0 k_i^2 = -1, \quad i = 1, \ldots, n \tag{6.5}$$

we see that the twisted $\hat{c} = 52$ trees in Eq. (4.1) are nothing but a new factorization of the trees of the ordinary untwisted open NN string [30] at $c = 26$!

The same NN trees are found as well for multiple emission from the twisted open string at $\xi = 0$, where the half-integer-moded DN coordinates in Eq. (3.1c) are trivially suppressed. These evaluations also exhibit the expected [17-19] no-ghost theorem (including the decoupling of zero-norm states) for the twisted $\hat{c} = 52$ sector of this orientation orbifold.

7. Open- and Closed-String Regge Slopes

As a byproduct of our computation

$$\text{closed}\,(c = 26) \;\rightarrow\; \text{twisted open}\,(\hat{c} = 52) \;\simeq\; \text{untwisted open}\,(c = 26) \tag{7.1}$$

we also obtain a simple new derivation of the ratio of the Regge slope α'_c of the original untwisted closed string to the slope α'_0 of the resulting untwisted

open string. The key is the appearance of the *same* [12,13,15] *dimensionless momenta* $\{T\}$ in both the untwisted and the twisted sector of the orientation orbifold [5].

In fact, we need only combine Eqs. (2.2) and (6.5) to find the correct relation [31] between the slopes

$$\sqrt{\alpha'_c}\, k = T = \sqrt{\frac{\alpha'_0}{2}}\, k \tag{7.2a}$$

$$\longrightarrow \quad \alpha'_0 = 2\alpha'_c \tag{7.2b}$$

where $\alpha(s) = \alpha(0) + \alpha' s$ is the form of either leading trajectory. Here I have assumed only that the dimensionful momenta $\{k\}$ are the same for both string types – which is required by momentum conservation in any open-closed string interaction. I also emphasize that our computation involves going off-shell

$$\text{closed:} \quad T^2 = -2 \quad \longrightarrow \quad \text{open:} \quad T^2 = -\frac{1}{2} \tag{7.3}$$

in order to pass between the ground states of the two strings. This reflects the fact that the orientation-orbifold construction {untwisted closed string $\rightarrow$ twisted open string} is fundamentally conformal-field-theoretic.

8. Conclusions

We have confirmed at the interacting level the conclusion reached for the physical spectrum in Ref. [19]: As a string system, the simple orientation orbifold

$$\frac{U(1)^{26}}{H_-}, \quad H_- = (1,\, \tau_1 \times (-1)) \tag{8.1}$$

is equivalent to the archetypal orientifold

$$\sigma = 0: \quad \text{unoriented closed string at } c = 26 \tag{8.2a}$$

$$\sigma = 1: \quad \text{twisted open string at } \hat{c} = 52 \tag{8.2b}$$

$$= \text{ordinary } NN \text{ string at } c = 26.$$

[5] This phenomenon is quite general in the orbifold construction of twisted sectors from the untwisted sector, so that e.g. the untwisted representation matrices $\{T\}$ of Lie g appear in the twisted representation matrices $\{\mathcal{T}(T)\}$ of the twisted sectors of WZW orbifolds [6-11] and WZW orientation orbifolds [12,13,15]

that is, the ordinary open-closed bosonic string system. Indeed, the evaluation of the twisted trees in Sec. 6 has sharpened the equivalence even at the spectral level – showing now the correct decoupling of null physical states. As a bonus, our unconventional formulation of the conventional system also provided a new derivation of the ratio (7.2b) of open- to closed-string Regge slopes.

This clarifies the somewhat mysterious relation between the orientation orbifold (8.1) and the orientifold – both of whose open-string CFT's have been differently associated to division by the world-sheet orientation-reversing automorphism $(-\tau_-)$. The resolution is that, *as mass-shell string theories*, these particular two open-string sectors are identical, even at the interacting level.

Although we have found equivalence in this case for the spectrum and mass-shell emissions at the string boundaries, I emphasize that the orientation-orbifold picture remains *qualitatively different* than the conventional picture in other regions of the theory, including

$$\bullet \text{ the bulk } 0 < \xi < \pi \text{ of the twisted } \hat{c} = 52 \text{ open string} \tag{8.3a}$$

$$\bullet \text{ off mass-shell} \tag{8.3b}$$

$$\bullet \text{ as an open-string orbifold CFT} \tag{8.3c}$$

where the *half-integer modeing* of standard orbifold theory persists for the open-string sector of the orientation orbifold.

Perhaps our most important conclusion however is the subtitle of this paper : "Orientation orbifolds *include* orientifolds". This statement follows from the identification (8.2) and the work of Refs. [17-19] where it is shown that, without appending any Chan-Paton structure, there are *many other orientation-orbifold string systems* $U(1)^{26}/(\mathbb{Z}_2(\text{w.s.}) \times H)$ with higher fractional modeing $\{n(r)/\rho(\sigma)\}$. These constructions, with many twisted closed- and open-string sectors, are generically *inequivalent* [19] to untwisted string systems and should be further examined for consistency at the interacting level.

Towards this, one should bear in mind that the twisted closed-string sectors of each orientation orbifold form an essentially-ordinary space-time orbifold [1-15] at $c = 26$, while each $\hat{c} = 52$ twisted open-string sector has an equivalent but unconventionally twisted $c = 26$ spectral description [19] in terms of the unconventional matter-field fractions $\{2n(r)/\rho(\sigma)\}$. For exam-

ple, the orientation-orbifold CFT's

$$\left(1,\,\omega_3,\,\omega_3^2\,;\;\tau_-,\,\tau_-\times\omega_3,\,\tau_-\times\omega_3^2\right),\quad \omega_3^3=1 \tag{8.4a}$$

$$\left(1,\,\omega_4^2\,;\;\tau_-\times\omega_4,\,\tau_-\times\omega_4^3\right),\quad \omega_4^4=1 \tag{8.4b}$$

contain $1/3-$ and $1/4-$integer modeing respectively, though the $\hat{c}=52$ open strings of the latter have an equivalent half-integer-moded $c=26$ spectral description. It will be interesting in particular to understand whether the equivalent $c=26$ spectral description of the general orientation orbifold-string can be extended (as seen for our special case here) to an equivalent but generically-new $c=26$ description of all these systems at the interacting level. Some further remarks on the general-free bosonic orientation orbifold are included in the Appendix.

Appendix A. Reducible Vertex Operators

For the particular orientation orbifold studied in this paper, Ref. [15] gives a set of twisted vertex operators $\{\hat{g}(\mathcal{T})\}$ which have an additional 2×2 matrix structure relative to the ones studied here. The matrix vertex operators are easily obtained from the vertex operators $\hat{\hat{g}}_+$ in Eq. (3.5) by the substitution

$$\hat{x}_{1\mu u} \to \hat{x}_{1\mu u}\tau_u, \quad \bar{u} = 0,1 \tag{A.1a}$$

$$\hat{\hat{g}}_+(T) \;\to\; \hat{g}(\mathcal{T}) \propto \;: e^{-\frac{i}{2}\hat{x}\cdot\mathcal{T}} : \tag{A.1b}$$

where τ_0, τ_1 are respectively the 2×2 unit matrix and the first Pauli matrix. In this notation, the twisted representation matrices $\{\mathcal{T}\}$ have the form $\mathcal{T}_{\mu u} = T_\mu \tau_u$ where $\{T\}$ are the dimensionless closed-string momenta of the text.

Correspondingly, the commutators of the matrix emission operators $\hat{g}(\mathcal{T})$ at $\xi = \pi$ with the twisted currents $\hat{J}$ and extended Virasoro generators $\hat{L}$ are obtained from those in Eq. (3.9) by the additional substitutions

$$\hat{g}_+(T) \to \hat{g}(\mathcal{T}), \quad \hat{J}_{1\mu u} \to \tau_u \hat{J}_{1\mu u}, \quad \hat{L}_u \to \tau_u \hat{L}_u. \tag{A.2}$$

Twisted matrix tree graphs $\hat{A}(\{\mathcal{T}\})$ can also be constructed as in Eq. (4.1), now using matrix multiplication of $\hat{g}$'s with all $\tau_u^{(i)} = \tau_u$. Then one finds that the following matrix gauges

$$\hat{W}_u(m+\tfrac{u}{2}) = e^{-i\pi(m+\frac{u}{2})}\tau_u \hat{L}_u(m+\tfrac{u}{2}) - \Big(\hat{L}_0(0) + m + \tfrac{u}{2} - \hat{a}_2\Big) \tag{A.3}$$

are operative in the matrix trees, leading to the same extended physical state conditions (5.4b). Explicit evaluation of the matrix trees gives

$$\hat{A}_n(\{\mathcal{T}\}) = \tau_0 \hat{A}_n(\{T\}) \tag{A.4}$$

that is, two copies of the NN amplitudes $\hat{A}_n(\{T\})$ obtained in Eq. (6.4).

This result can be understood as reducibility of the matrix vertex operators $\hat{g}(\mathcal{T})$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad U\tau_1 U^\dagger = \tau_3 \tag{A.5a}$$

$$U\hat{g}(\mathcal{T})U^\dagger = \begin{pmatrix} \hat{g}_+(T) & 0 \\ 0 & \hat{g}_-(T) \end{pmatrix} \tag{A.5b}$$

where $\hat{g}_+(T)$ (or $\hat{\hat{g}}_+$ for all ξ) is the vertex operator of the text. The vertex operator $\hat{g}_-(T)$ differs from $\hat{g}_+(T)$ only by a sign reversal $\hat{x}_{1\mu 1} \to -\hat{x}_{1\mu 1}$ of the NN component of the $\hat{c} = 52$ string, which reproduces Eq. (A.4) because all vertex-operator contractions are pairwise.

In the computations of the text, I have kept only one irreducible component $\hat{g}_+$, which is equivalent to the replacement $\tau_u \to 1, \mathcal{T} \to T$ in the matrix vertex operator $\hat{g}(\mathcal{T})$. Certainly, this choice is sufficient to satisfy ordinary open $\leftrightarrow$ closed string duality in this case. More physically, the prescription may be understood as division by the symmetry $\hat{g}_+ \leftrightarrow \hat{g}_-$, which is a residual form of world-sheet parity [12,13,15] in this basis.

Refs. [12,15] give mode expansions for the coordinates $\hat{x}_\sigma$ and vertex-operator equations for the twisted vertex operators

$$\hat{g}_(\mathcal{T}, \sigma) \propto \ : e^{i\hat{x}_\sigma^{n(r)\mu u} \mathcal{T}_{n(r)\mu}(T,\sigma) \otimes \tau_u} : \tag{A.6}$$

in open-string sector σ of the general free-bosonic orientation orbifold (1.1). The explicit form of the twisted representation matrices $\{\mathcal{T}(T, \sigma)\}$ is given in Eq. (2.20d) of Ref. [12], where the quantities $\{T\}$ in this application are the same dimensionless critical closed-string momenta of the text. Each of these vertex operators has the same 2×2 reducible matrix structure discussed here and, dividing by world-sheet parity, I would speculate that the same prescription $\tau_u \to 1$ will be sufficient to satisfy open $\leftrightarrow$ closed string duality in the general case.

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